Behavior of a ball on the surface of a rotating disk

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Energy dissipation motion of a ball on a rotating disk (a turntable) has been considered. It is shown that the motion consists of two consequent stage - the motion towards the disk center along a cardioidic curve and the motion from the center along an unreeling helix. The limits of applicability of the results obtained are analyzed and qualitative comparison with experimental data is carried out.

I. INTRODUCTION

Anyone can easily repeat the experiment which yielded the unexpected results that inspired our interest in this subject. You need a turntable (we propose a record player) and a small plastic ring with a diameter of about 20 mm (see Fig. 1). Switch the player on and put the ring on the rotating turntable, so that it rolls near the edge of the turntable with zero velocity center with respect to the laboratory. Let us try to predict its future behavior. It seems natural to expect that the ring will rotate around a fixed point within some time period, then that friction will force it to follow the rotation of the turntable and finally centrifugal force will cast it away from the turntable.

Now start the experiment and you will see a mysterious thing. If you are skillful enough to place the ring on the turntable such that it starts rolling over the turntable, rather than falling off, you will see that in spite of your expectations it moves towards the center.

Now try to replace the ring with a ball. You will see here that there is almost no difference in behavior between a ball and ring. Since a theoretical investigation of the motion of a ball is far simpler than for a ring, we prefer to analyze this case.

The problem of the motion of a ball along a steady surface is well understood.[1-4] The motion of a ball on rotating surfaces is considered in Refs. 5–8. To those who are interested in the history of the problem we recommend reading Ref. 9. On an ellipsoidal surface the motion of a ball is described in Ref. 10. Equations describing the motion of a body rolling with dissipation are derived in Ref. 2.

II. THE FIRST ATTACK: WHETHER A KINEMATICS CONSTRAINT EXPLAINS THE PHENOMENA

After you have experimented enough with different balls, rings, and coins you note that the observed motion is a superposition of at least three different kinds of motion.

(1) The body continues to roll, keeping the direction of the angular velocity vector approximately constant.

(2) The body exhibits fast oscillations (whose frequency is of the order of the disk rotation frequency) around its initial position.

(3) The center of the oscillations moves (rather slowly) towards the disk center.

The first point can be easily understood due to the analogy with a gyroscope but the two remaining points are not so evident.

Let Ω be the angular velocity of the rotating disk, $\tilde{R}$ and $\tilde{m}$ be the radius and the mass of the ball, respectively. We will assume that the ball rolls without slipping on the disk surface. Dimensionless variables (Ω, R, m) will be used, where $\tilde{R}$, $\tilde{m}$, and $\tilde{\Omega}^{-1}$ are taken as the units of length, mass, and time (the tilde will indicate dimensional variables). Thus, for example, $\Omega^{-1}=R=m=1$.

Since the ball is rolling over a plane, the location of its center may be given by a two-dimensional vector, r, from

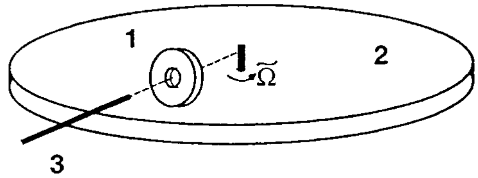

Fig. 1. A ring (1) rolling without slipping on a surface of the rotating disk (2). The axis (3) is already removed.

the turntable center to the point of contact (Fig. 2). We introduce the Cartesian (orthogonal) coordinate system (x,y,u) with its origin at the disk center, the direction of the u axis is the same as the direction of the angular velocity Ω of the turntable. We may compose one complex number out of two coordinates of any vector in (x,y). Henceforth, let $z=r_x+ir_y$, $\omega=\omega_x+i\omega_y$ be complex representations of the position and the angular velocity of the ball by wmplex numbers. (Note that the third component of ω plays no role for determining the ball positions.) Thus the problem will be solved if we determine the time dependence of those two numbers.

If the turntable is horizontal, the only force acting upon the ball in the (x,y) plane is the force of friction, assuming the absence of slipping. We will analyze a more general situation if we assume the presence of an additional force, $\mathbf{F}_0$ (the result of the support reaction and the gravitational force) arising if the turntable is tilted. Newton's law says that

$$m\mathbf{r}''=\mathbf{F}_0+\mathbf{F}_{fr}, \tag{2.1}$$

Expressing Eq. (2.1) in terms of dimensionless complex variables, we obtain

$$z''=F_0+F_{fr}. \tag{2.2}$$

Another equation is given by the law of angular momentum. Because the only force producing a nonzero angular momentum with respect to the center of the ball is $\mathbf{F}_{fr}$, we get

$$J\omega'=\mathbf{R}\times\mathbf{F}_{fr}. \tag{2.3}$$

Here, J is the moment of inertia of the ball (equal to $\frac{2}{5}$ for a solid ball and $\frac{2}{3}$ for a hollow sphere, if the mass and radius of the ball have been taken equal to unity). One can easily show that the vector product of $\mathbf{e}_u$ and any vector in the (x,y) plane is equivalent to multiplication of a complex number $z=x+iy$ by i, the imaginary unit. In terms of complex variables Eq. (2.3) can be rewritten in the form

$$J\omega'=iF_{fr}. \tag{2.4}$$

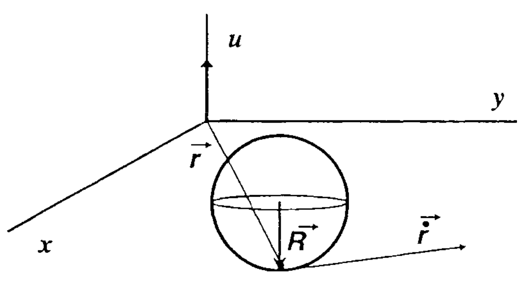

Fig. 2. The coordinate system.

The value of the friction force is determined by the no-slip condition, which means that the turntable and the ball have equal velocities at the contact point. The velocity of the turntable at this point is simply $\Omega\times$r, or $-iz$ in terms of complex variables, while the velocity of the ball is equal to a sum of the center velocity v and the relative velocity due to rotation, $\omega\times\mathbf{R}$. As a result we get

$$\Omega\times\mathbf{r}=\mathbf{v}+\omega\times\mathbf{R} \tag{2.5}$$

or

$$\omega=z+iz'. \tag{2.6}$$

Since the angular velocity ω is not generally a total derivative of any coordinate, this relation cannot be integrated, i.e., the relation (2.1) is not a holonomic one.[1–3]

Now we have a sufficient number of equations [Eqs. (2.2), (2.4), and (2.6)] to eliminate the unknown quantities ω and F_{fr}, and obtain an ordinary differential equation for $z(t)$. If $\xi=J/(J+1)$, $f_0=F_0/(J+1)$, the following equation

$$z''-i\xi z'=f_0 \tag{2.7}$$

holds due to Eqs. (2.2), (2.4), and (2.6).

In the case of a horizontal turntable, Eq. (2.7) provides the conservation law,

$$z'-i\xi z=\text{const}. \tag{2.8}$$

The left-hand side of Eq. (2.8) gives an expression for the angular momentum with respect to the contact point, and as expected it is conserved. The solution of Eq. (2.8) is given by

$$z=\rho+ae^{i\xi t}, \tag{2.9}$$

where constants ρ and a are determined by the initial conditions

$$z(t=0)=z_0,\quad z'(t=0)=v_0. \tag{2.10}$$

Thus without an additional force $\mathbf{F}_0$, the friction force causes the ball to behave like a charged particle in a magnetic field,[10] i.e.,to perform a uniform circular motion with a constant absolute velocity, $|z'|=v_0$, along a circle with radius $|a|$ and center at the point $z=\rho$. Note that $|a|=|v_0|/\xi$ does not depend on the initial position of the ball, but only on the value of the initial velocity.

Let us now return to the case of nonzero f_0. The right-hand side of Eq. (2.7) vanishes after the substitution

$$z=\hat{z}+(if_0/\xi)t. \tag{2.11}$$

In this case $\hat{z}$ may be interpreted as a complex coordinate of the ball's center referred to a coordinate system moving with the velocity f_0/ξ in a direction perpendicular to the slope. The electromagnetic analogy may be extended, and we can say that the motion under study coincides with that of a charged particle in crossed electrical and magnetic fields.'

The obtained results are interesting and unexpected by themselves but they do not explain the centripetal motion observed in our experiment. This means that our model misses some important factors, the most significant of which is the neglect of rolling friction. The next section is devoted to the effect of rolling friction on the ball's motion.

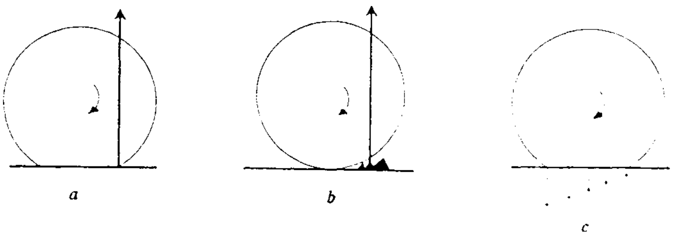

Fig. 3. The reasons for rolling friction force arising from: (a) deformation, (b) micro-obstacles, and (c) adhesion.

III. THE SECOND ATTACK: THE EFFECT OF ROLLING FRICTION

In the previous section we have introduced the force of friction that ensured the absence of slipping. Obviously this force vanishes if a ball's center performs a uniform straight-line motion along a fixed plane surface.

At the same time our experience says that the ball rolling along the surface will stop sooner or later and thus there must be some force resisting the motion. We will refer to this force as a rolling friction force **(RFF). RFF** arises due to the deformation of the ball, and surface micro-obstacles along the ball path as well as due to adhesion. Figure **3** demonstrates schematically how RFF arises.

It is suggested by our qualitative consideration that **RFF** has no horizontal components, but only produces the moment directed counterwards to the angular velocity of the ball. Certainly, the value of the moment may depend on the absolute value of the angular velocity in a rather complicated way. Thus we can assume that the rolling friction may be chosen in the form

$$\mathbf{M}=-\alpha(\omega)\boldsymbol{\omega}, \tag{3.1}$$

where M is the angular moment of the **RFF** with respect to the center of the ball, $\boldsymbol{\omega}$ is angular velocity of the ball, and $\alpha(\omega)$ is a coefficient that may depend on the absolute value of $\boldsymbol{\omega}$.

Certainly, we do not pretend to find an expression for $\alpha(\omega)$, because it **will** depend on a lot of parameters defined by both the ball and the surface material, their machining, and so on. Still we will concentrate on the simplest case $\alpha(\omega)=\text{const}$, because it admits an analytical solution and as it will be shown is in a good correspondence with experiment.

According to the above reasons, we placed an additional term into the right-hand side of Eq. **(2.4)** and got

$$J\omega'=iF_{fr}-\alpha\omega. \tag{3.2}$$

Using **Eq. (3.2)** along with Eqs (2.2) and (2.6), which remain valid regardless of the introduction of **RFF,** we obtain

$$z''+(\eta-i\xi)z'-i\eta z=f_0, \tag{3.3}$$

where $\eta=\alpha/(J+1)$. We may eliminate f_0 from the last equation as we did from Eq. (2.7) by the simple substitution $\hat{z}=z-if_0/\eta$.

One may *be* **confused** by the fact that the force f_0 results in a shift of the coordinate system origin rather than a motion of the coordinate system, as in Eq. (2.7). [Note that **Eq. (2.7)** is a particular **case** of Eq. **(3.3)** where $\eta=0$]. Indeed there is no contradiction: if η tends to zero,

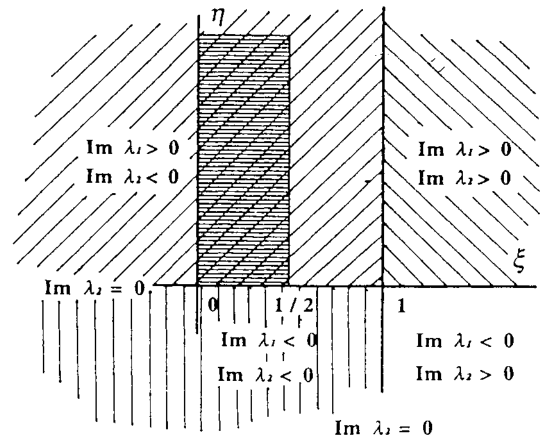

Fig. 4. Regions of constant stability-type of the ball motion.

the shift if_0/η will tend to infinity, and we may treat the uniform motion ensured by the force f_0 as a motion towards the infinitely shifted new center. Note that the shift and motion directions correspond to each other.

If $f_0=0$, Eq. **(3.3)** becomes a homogeneous linear differential equation and its solution may be expressed in terms of the roots of its characteristic equation

$$\lambda^2-(\xi+i\eta)\lambda+i\eta=0 \tag{3.4}$$

as follows

$$z=C_1\exp(i\lambda_1 t)+C_2\exp(i\lambda_2 t). \tag{3.5}$$

Here, C_1 and C_2 are constants to be determined from the initial conditions. Each term in Eq. **(3.5)** describes a motion along a reeling or unreeling helix according to the sign of the imaginary part of the corresponding root.

Due to the Viette theorem Eq. **(3.4)** is equivalent to the system

$$\lambda_1+\lambda_2=\xi+i\eta, \tag{3.6}$$

$$\lambda_1\lambda_2=i\eta. \tag{3.7}$$

Since the roots depend on ξ and η continuously, an imaginary part may change the sign under variation of ξ and η only after taking zero value.

Let $\operatorname{Im}\lambda_1=0$. Then the real **part** of Eq. **(3.7)** gives us

$$\operatorname{Re}\lambda_1\operatorname{Re}\lambda_2=0,$$

and we conclude that either $\operatorname{Re}\lambda_1=0$ and thus $\lambda_1=0, \lambda_2=\xi$, $\eta=0$, or $\operatorname{Re}\lambda_2=0$ and in this case $\lambda_1=\xi=1$ and $\lambda_2=i\eta$. The explicit form for the roots of Eq. **(3.4)** is

$$\lambda_{1,2}=(\xi+i\eta\pm\sqrt{(\xi+i\eta)^2-4i\eta})/2. \tag{3.8}$$

Note, that $\lambda_1\to 0, \lambda_2\to\xi$, with $\eta\to 0$. After all we **can** divide the (ξ,η) plane according to the signs of the imaginary parts of λ_1 and λ_2 (Fig. **4).** The figure shows that different pairs of ξ and η **may** result in different characters of the motion. Nevertheless the possible range of variation for those parameters is restricted by

$$\eta>0,\quad \frac{1}{2}>\xi=\frac{J}{J+1}>0.$$

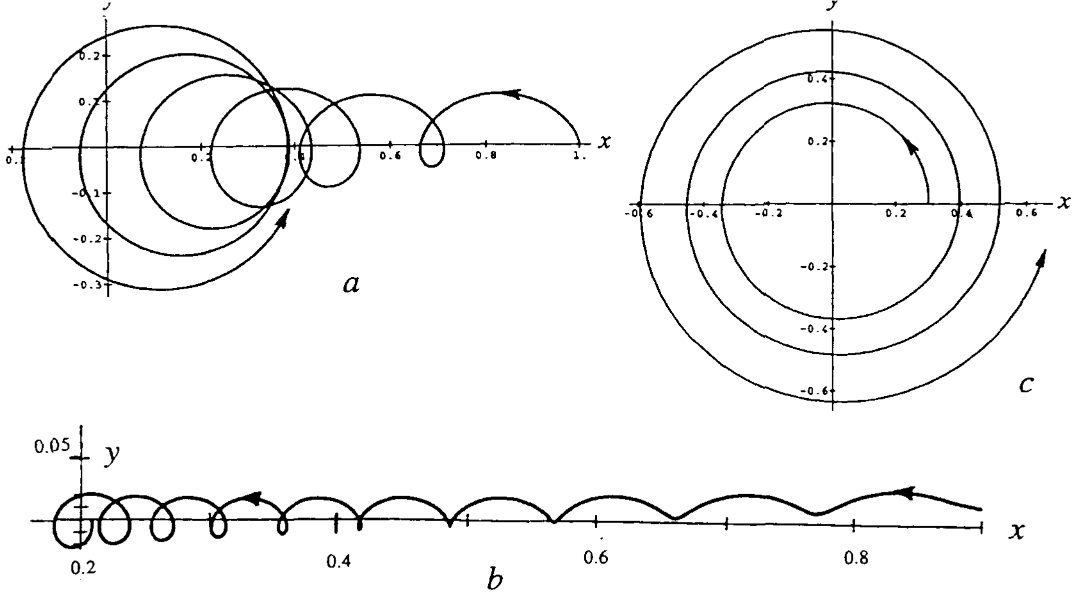

Fig. 5. The path of the ball in the laboratory system of frame for (a) $\eta=0.005$, $v_0=0$; (b) $\eta=0.002$, $v_0=0.004$; (c) $\eta=0.005$, $v_0=0.086$.

This region is shaded by horizontal lines in Fig. 4 from which one can see that in the allowed parameter range we **will** always have $\mathrm{Im}\,\lambda_1>0$ and $\mathrm{Im}\,\lambda_2<0$.

Such a solution means that for the determination of stability it is sufficient to treat the case of small nonzero rolling friction coefficients as $0<\eta<1$. Using this approximation considerably simplifies the formulas and describes the real situation with reasonable accuracy. The first order approximation may be given in this case by

$$\lambda_1=i\eta/\xi, \quad \lambda_2=\xi-i\eta/J. \tag{3.9}$$

In the experiment discussed at the beginning of this paper we suggested putting a ball or a ring on the turntable with zero initial velocity. If the initial point has a complex coordinate z_0 the solution will take the form

$$z=z_0[(1+i\eta\xi^{-2})e^{-t\eta/\xi}-i\eta\xi^{-2}e^{t\eta/J}e^{i\xi t}]. \tag{3.10}$$

The second term describes a helix unreeling away from the circle of radius $\eta\xi^{-2}$, while the first shows that the center of the helix moves towards the center of the turntable. Notice that this first term may be responsible for the observed motion towards the center.

Indeed at the beginning of the motion and during a time of about J/η, the first term of Eq. (3.10) is the dominant one due to the small value of η. Thus during a considerable time the ball will move (slowly) towards the center, performing oscillations with an increasing amplitude due to the second term. As for the ring, it may even stop its motion by falling in the center of the disk. While moving near the center of the disk, it rotates slowly and its behavior looks like that of a coin being thrown on a table. The ball, on the other hand, will sooner or later move along an unreeling helix and its motion does not look as strange as that of a ring. It is worth mentioning that if we restore the usual units in the expression for the characteristic time of unreeling of the helix, we find that it does not depend on the angular velocity Ω of the turntable. In fact

$$\eta=\frac{\tilde{\eta}}{\tilde{m}\tilde{R}^2\tilde{\Omega}} \tag{3.11}$$

Here, a tilde over a variable indicates to us that we are dealing with the dimensional value. Hence, from Eq. (3.11) the dimensional time of the change of stages t_0 is

$$\tilde{t}_0=t_0\tilde{\Omega}\approx\frac{\tilde{m}\tilde{R}^2}{\tilde{\eta}}. \tag{3.12}$$

Returning to Eq. (3.11) we see that the limit $\eta\to 0$ is the limit of both small rolling friction and a large angular velocity of a turntable. Thus we may decrease the parameter η by increasing the angular velocity of the turntable in order to see the motion describe by Eq. (3.10). One can easily prove that an increase in Ω influences only the last exponent of Eq. (3.10), i.e., the frequency of the ball rotation along the circle with the radius $|a|$. The drift velocity to the disk center $z_0\alpha/J$, the rate of increase of the radius, and the time t_0 of switching between the stages do not depend on the angular velocity of the disk $\tilde{\Omega}$ at all. Thus even if the disk rotation rate changes with time the system behavior does not differ strongly from what has been described above.

IV. DISCUSSION

Someone who knows the history of the problem of the ball on the turntable (for example, from Refs. 6 and 9) may ask a question: Why have not the effects and results, mentioned in our work, been discussed in the literature earlier? The answer is simple - in order to watch the motion of the ball towards the center, it is necessary to experiment with a "bad" system, where rolling friction is present. For example, the system will be "bad" if the ball or the disk is made from a "soft" material.

In laboratory installations only "good" materials are used. As for our experiments, they were made with a record player and an old record which is good for nothing! So it is easier to make a "bad" system than a "good" one, and it can even be made in a home lab. Typical trajectories which can be seen in the experiment are similar to the ones demonstrated in Figs. 5(a)-5(c), where the results of the calculation according to Eq. (3.5) with different values of system parameters are given.

The two stages of the motion are seen in Fig. 5(a). The initial velocity of the ball is small. When the rolling friction coefficient η is small enough, the ball can move towards the disk center almost without oscillations [Fig. 5(b)]. When the coefficient η increases ($\eta > 0.01$) or the initial velocity increases, the first stage practically vanishes and the amplitude of the oscillations grows constantly. With a large coefficient of rolling friction [Fig. 5(c)] the motion goes to a more pure case of an unreeling helix.

Although all results are obtained for a body with spherical symmetry, the experiments show that the motion of bodies with cylindrical symmetry has the same character if the following conditions are met. First. the moment of inertia with respect to the axis of symmetry has to be an appropriate size in comparison with the other moments of inertia. Second, the surface of the rolling body must touch the disk surface at only one point. The latter can be provided even for a thin cylinder (a coin) put on the disk surface at a small angle.

All the described kinds of motion of the body can come to an end due to a number of reasons.

(i) Moving along the unreeling helix, the body will reach the boundary of the disk,

(ii) The motion of a nonspherical (ringlike) body while passing the center at a small speed can be transformed in a stage not described by the given theory,

(iii) The regular motion can be destroyed when the acceleration $|z''|$ of the ball exceeds the maximum static friction force

$$kg \geqslant |F_{fr}|, \tag{4.1}$$

where g is a free-fall acceleration and k is a static friction coefficient (or the difference $|F_{fr}| - |F_0|$ for a disk with a slope, dimensionless units). In this case a slip begins. Thus, for a motion along a circle without rolling friction the static friction coefficient k must satisfy

$$k > \frac{\tilde{v}_0 \xi}{\tilde{m}\tilde{R}\tilde{\Omega}\tilde{g}}. \tag{4.2}$$

(iv) The roughness of the disk surface is one more reason for the **destruction** of the regular motion. Thus, with a high enough rate $\tilde{\Omega}$, even smooth defects of a disk surface can result in an essential change in the normal pressure of the ball on the surface. This is equivalent to decreasing the maximum friction force at this point of the disk surface. Given the surface profile $h(x,y)$ the appropriate calculation can be performed. **A** small asymmetry of the body leads to similar behavior.

Finally, we note that our experiment qualitatively confirms the theoretical results. The experiments were performed with a 0.2-m disk with a variable frequency of rotation (0.5–2 s^{-1}). Metal balls (3-mm diameter) and metal rings (6-20 mm diameter, 1–3 mm width) were used. The following phenomena described by the theory were observed:

motions of a ball and of a ring are similar;

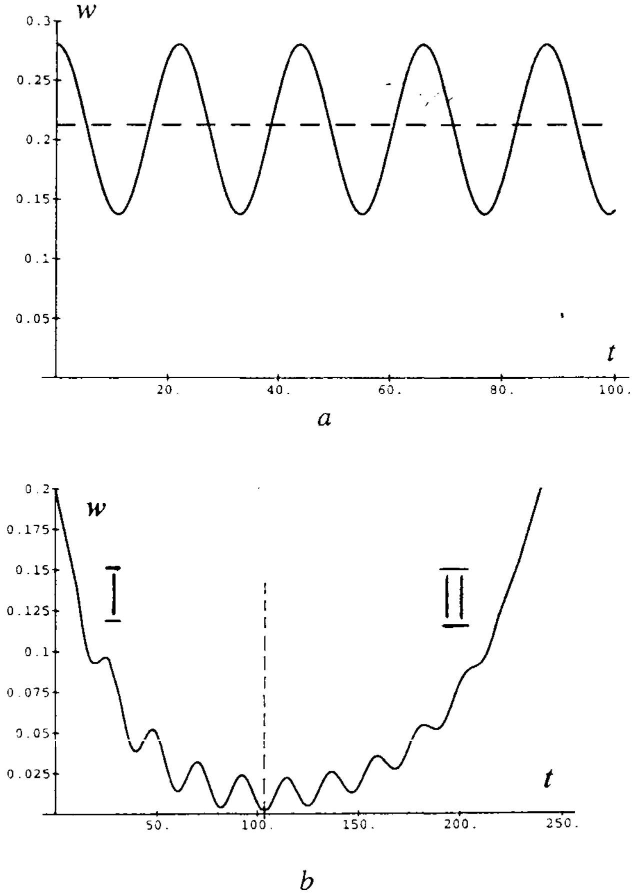

Fig. 6. The time dependence of the kinetic energy of the ball for (a) $\eta = 0$ and (b) $\eta = 0.005$.

when the initial velocity of the body is zero it starts to drift towards the disk center;

the influence of the initial conditions increases with the increasing of the disk rotation velocity;

the frequency of the oscillations increases with increasing disk rotation velocity;

the character of the path coincides with that obtained theoretically.

A rare person may wonder why the kinetic energy of the ball is not conserved while it performs the described motion. In fact, the ball loses or gains energy due to the interaction with the turntable. Nevertheless we found it interesting to investigate the behavior of the ball's energy. The kinetic energy of the ball consists of an energy of linear motion and rotational energy. It is given by

$$W = \frac{|z'|^2}{2} + \frac{J|\omega'|^2}{2} = \frac{1}{2}(|z'|^2 + J|z + iz'|^2). \tag{4.3}$$

In order to obtain an explicit expression for W as a function of time we have to take z as a function of the time investigated above, calculate its derivative, and substitute $z(t)$ and $\dot{z}(t)$ into Eq. (4.3). Taking $z(t)$ from Eq. (2.9), we get

$$W(t) = \tfrac{1}{2}[(a\xi)^2 + \rho^2 + a^2(1-\xi)^2 + 2\rho a(1-\xi)\cos \xi t] \tag{4.4}$$

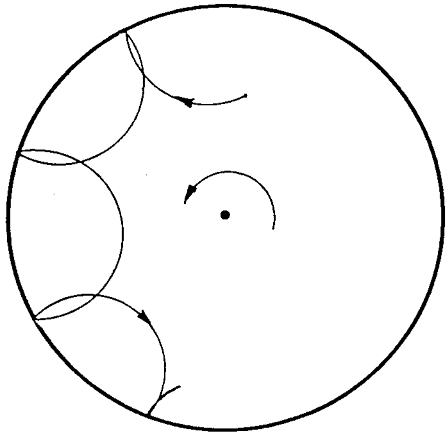

Fig. 7. Ball motion on the turntable with a board.

and see that the ball energy oscillates in period with its circular motion around some nonzero mean value [see Fig. 6(a)].

A substitution of Eq. (3.10) into Eq. (4.3) in an attempt to treat an effect of RFF on the behavior of the ball's energy results in a much more complicated expression than Eq. (4.4). That is why we present only a graph drawn by a computer for this case [Fig. 6(b)]. Though the graph appears to be complicated, it admits a qualitative explanation. First of all, the figure depicts two stages of the ball's motion. While the ball moves towards the center its energy decreases though it is subjected to oscillations which have the same origin as the oscillations in the case of zero RFF. The second monotonically increasing part of the graph suggests that the ball has started its motion along the unreeling helix and is accelerated by both friction and "centrifugal" force. We cannot provide any simple explanation for the intermediate region. Note only that the energy may nearly vanish in this region. It may happen that the ball passes the center of the turntable with a very small velocity and almost stops there. At this moment the difference between a ball and a ring becomes significant. (In fact it is insignificant only if the angular velocity of the ring is large with respect to its axis.) A ball will cross the center and start an unreeling motion while a ring will fall when it loses its energy.

Doing experiments with a ball and a turntable, we discovered one final point of interest. If there is a board at the edge of the turntable, the ball will move along the board in the direction opposite to the rotation of the turntable. Now we have enough information to explain this phenomenon. We remember that the ball has to perform a uniform circular motion, but if it strikes the board it will reflect and continue its motion along another circle (see Fig. 7). We may treat the observed motion as a limiting case of the one described.

V. CONCLUSION

The theory presented here provides a description of all the observed phenomena. It shows the crucial role of rolling friction for the motion of the body of rotation towards the center of the rotating disk. The theory predicts independence of the following parameters on the rotation velocity: characteristic time of motion, velocity of drift towards the center, and the rate of increase of the radius of fast oscillations. The predicted properties are observable by experiment.

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